A Geology Simulation Problem in Grid Refinement Methods

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Abstract - Grid Refinement Method has been developed to solve linear systems generated from partial differential equations. Simulation of a geological problem by using the Grid Refinement Method along with iterative methods is presented in the paper. Accuracy and efficiency of the Grid Refinement Method is investigated for comparison with the solutions obtained by uniform grid approach.

Index Terms - Grid Refinement Method, linear system, partial differential equation, iterative methods, stopping procedure, Successive Over Relaxation Method, matrix decomposition, Dirichlet boundary conditions

1. INTRODUCTION

Grid Refinement methods [8, 9] have been used to solve large linear systems developed from partial differential equations. In this research, we study a problem that arises from the traveling of groundwater flow [1, 2] and the method to estimate how fast the contaminant disperses around the sinkhole in geological sciences. Suppose there is a rectangular domain where the boundary conditions are given and the initial contamination value at the sinkhole is also known, we would like to know the values in the region around the sinkhole. Researchers have been using the model "MODFLOW" to solve the problem by a two-step procedure. The major drawback of this two-step method is the computational time and the inconvenience of the interpolation process due to the two linear systems generated from procedure, as well as considerably more computational time to solve the systems.

Grid Refinement methods improve the above method by generating only one system of linear equations, which contains the information of all points that we are interested in, with the consideration of the treatments of the interface boundary points. In the existing two-step scheme, the approximations for interface boundary points are obtained by the interpolation technique, while the grid refinement method uses a simple "modified" finite difference scheme.

2. GRID REFINEMENT METHOD

In this chapter, we describe a grid refinement method for solving a partial differential equation of the form:

$$A(x,y)u_{xx} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y) \qquad (2.1)$$

where A,C,D,E,F are functions of x and y, with Dirichlet boundary conditions[11] on a rectangular region.

The basic idea of the grid refinement method is to decompose the original spatial domain into several sub-domains. For simplicity, we describe the method using two subdomains, namely interested domain and less interested domain. The coarse grids are put on the less interested domain; therefore it is also referred to as coarse grid domain. And the fine grids are put on the interested domain which is then also referred to as fine grid domain. Once the grids are placed, one linear system is generated. Then we are able to solve both subdomains simultaneously. We note that the fine grid subdomain could be formed in a rectangular shape or in other shapes, such as L shape or circular shape. In this research, we focus on rectangular shape. We also note that the fine grid subdomain could be placed anywhere within the original region to fit physical needs; again for simplicity, we assume in this research that the fine-grid subdomain is located in the center of the region, see Figure 2.1.

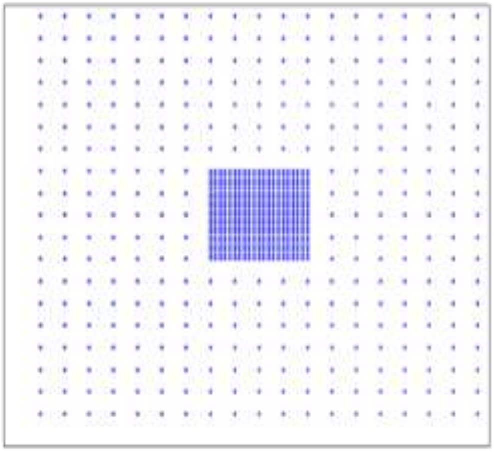

Figure 2.1

Once the linear system is generated according to the above grid pattern, we proceed with the solving the linear system in iterative methods[6, 7], such as Richardson's Method[10], Jacobi Method[3] and Gauss-Seidel [4]with Successive Over Relaxation Method[5]. The iterative algorithm produces a sequence of approximations $\{x^{(i)}\}$ to the exact solution $\bar{x}$ of the linear system (2.1), it is necessary to have a stopping procedure to determine whether the approximation is accurate enough to terminate the iterative procedure. In this paper, if the exact solution $\bar{x}$ is known, then it would be reasonable to accept the approximate solution $x^{(i)}$ if

$$\frac{\|x^{(i)}-\bar{x}\|_2}{\|\bar{x}\|_2} \le \varepsilon \quad (2.2)$$

where ε is a preset small tolerance, say 10^{-6}. We call the test (2.2) an exact stopping test.

3. NUMERICAL EXPERIMENT

We now attempt to solve a geology problem which is concerned with the groundwater flow and transport models. For the flow model which is referred to as MODFLOW [1, 2] is governed by the partial differential equation:

$$u_{xx} + u_{yy} = 0 \quad (3.1)$$

Over the region Ω= [0.5, 20.5] × [0.5, 20.5] with zero boundary conditions. However, there is a point sink source in the center of Ω. This point sink source can be treated as a water pollutant and the governing partial differential equation describes the diffusion of the water polluted and the concentration of the pollutant region is around the center. Therefore, we define the refined grid domain to be [8.5, 12.5] × [8.5, 12.5], which is the area of most interest that governs the spread of contamination of the pollutant.

It is set q = −10 at the central point as the sink source in our experiments.

The following table shows the computational time to obtain an accuracy of 10^{-6} to the linear system generated from the PDE using finite difference scheme with the uniform grid and with grid refinement scheme. Since we do not have the exact solution available, the stopping procedure (2.2) is used. In order to check the accuracy, we use the uniform grid as our benchmark solution. The computational time for the uniform grid scheme is ranging 3 to12 times more than the two-layer scheme.

Table 3.1 Computational time comparison for Geology Simulation 1

H	h	Size	Matrix Gen. Tim	Iter. Time	Total Time
1	1/2	417	1.0750	0.0220	1.0970
	1/2	1521	3.7960	0.0780	3.8740
1	1/4	625	1.6110	0.0340	1.6450
1/2	1/4	1729	4.3420	0.0900	4.4320
	1/4	6241	17.3020	0.6010	17.9030
1	1/5	777	2.4690	0.3880	2.8570
	1/5	9801	26.4450	1.2300	27.6750
1	1/8	1425	3.6860	0.0940	3.7800
1/2	1/8	2529	6.5050	0.1880	6.6930
	1/8	25281	69.2490	5.3770	74.6260
1	1/10	2017	5.7210	0.1750	5.8960
1/5	1/10	11041	30.9920	1.5310	32.5230
	1/10	39601	118.1860	9.6140	127.8000
1	1/16	4561	11.6850	0.5180	12.2030
1/2	1/16	5665	14.7190	0.6370	15.3560
	1/16	101761	369.4310	43.1830	412.6140
1/5	1/20	15921	47.7000	3.1690	50.8690
1/10	1/20	44481	202.1200	15.9320	218.0520
	1/20	159201	669.3290	80.3290	749.6580

Figures 3.1 & 3.2 show the contour graph of the solution over the entire region from the uniform grid with grid size h =1/5 and from two-layer scheme with H =1 and h =1/5, respectively. Both contour figures are very similar. We also extract the contour plots over the interested area in Figure 3.3 and Figure 3.4 for the uniform grid and two-layer scheme respectively. We notice that there is a little bit difference, but in general they are very similar. This similarity can be confirmed by the 3-D plots of the solution over the interested region for the uniform grid and two-

layer scheme given in Figure 3.5 and Figure 3.6 respectively.

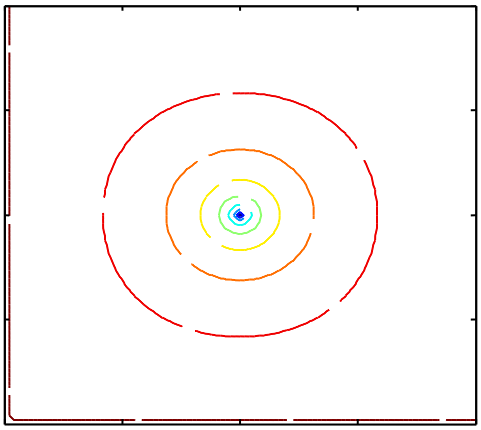

Figure 3.1 Contour figure of the entire region [0.5, 20.5] (H=1/5)

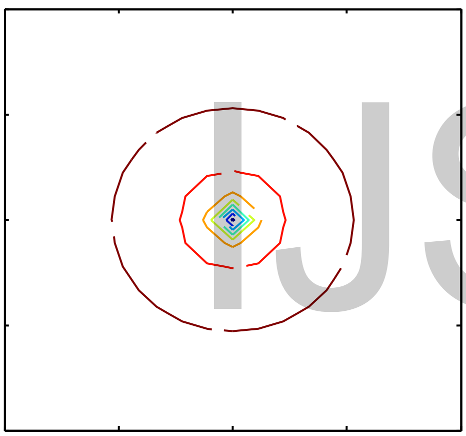

Figure 3.2 Contour figure of the entire region [0.5, 20.5] (H=1, h=1/5)

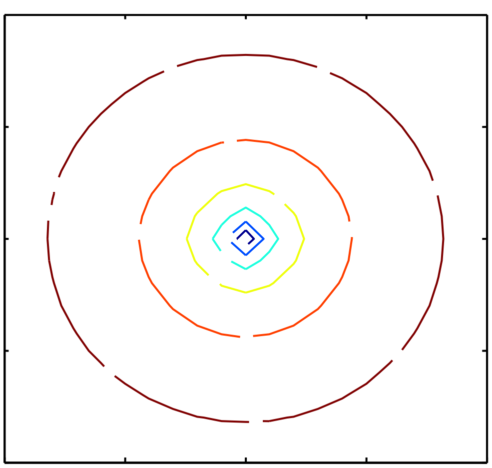

Figure 3.3 Contour figure of interested region [8.5, 12.5] (H=1/5)

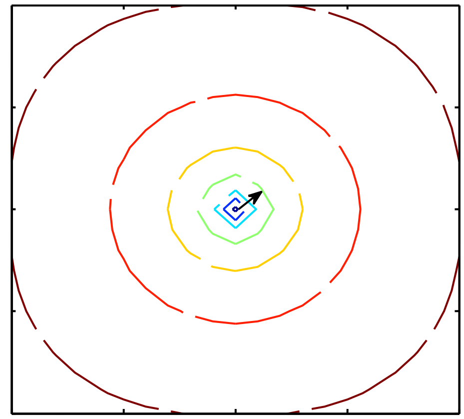

Figure 3.4 Contour figure of interested region [8.5, 12.5] (H=1, h=1/5)

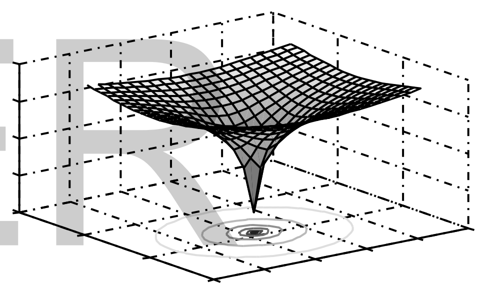

Figure 3.5
3-D figure of interested region [8.5, 12.5] (H=1/5)

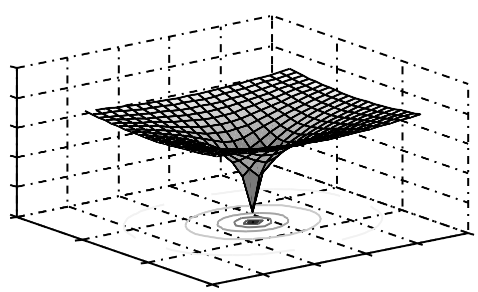

Figure 3.6 3-D figure of interested region [8.5, 12.5] (H=1, h=1/5)

4 CONCLUSIONS

In this paper, we have discussed the application of a grid refinement scheme for solving a partial differential equation over a rectangular domain with Dirichlet boundary conditions. A hypothetical geology problem is also conducted. The solutions obtained by the grid refinement scheme are very comparable to the solutions obtained by the uniform grid with a huge efficiency improvement.

REFERENCES

[1] S. Mehl and M. C. Hill, Development and evaluation of a local grid refinement method for block-centered finite-difference groundwater models using shared nodes, *Advances in Water Resources*, vol. 25, 487-511, 2002.

[2] S. Mehl and M.C. Hill, Three-dimensional local grid refinement for block-centered finite-difference groundwater models using iteratively coupled shared nodes: a new method of interpolation and analysis of errors, *Advances in Water Resources,* vol. 27, 899-912, 2004.

[3] I.N. Bronshtein and K. A. Semendyayev, Handbook of Mathematics, 3rd ed. Springer-Verlag, New York, 1997.

[4] H. Jeffreys and B.S. Jeffreys, Methods of Mathematical Physics, 3rd ed. Cambridge, Cambridge University Press, England, 1988.

[5] David M. Young, Iterative Solution of Large Linear Systems, Academic Press, 1971.

[6] Y. Saad and Yousef, Iterative Methods for Sparse Linear Systems, PWS Publishing Company, 1996.

[7] K. Atkinson, An introduction to numerical analysis (2nd ed.), John Wiley and Sons, 1988.

[8] Mai, T. Z., & Wu, L. Multi-layer Grid Refinement Method. *International Journal of Engineering Science and Technology*, 4(7), 3521− 3530, 2012.

[9] Mai, T. Z., & Wu, L. The successive over relaxation method in multi-layer grid refinement scheme. *Advances in Applied Science Research*, 4(2), 163-168, 2013.

[10] Stoer, Josef; Bulirsch, Roland. Introduction to Numerical Analysis (3rd ed.). Berlin, New York: Springer-Verlag, 2002.

[11] Cheng, A. and D. T. Cheng. Heritage and early history of the boundary element method, *Engineering Analysis with Boundary Elements*, vol. 29, 268–302, 2005